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MAGNETOHYDRODYNAMIC WAVES IN THE IONOSPHERE

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ABSTRACT*

The ten conditions which must be satisfied before simple magnetohydrodynamic (MHD) waves can propagate in a plasma are enumerated and discussed. Assuming that all but the tenth condition are satisfied throughout the ionosphere and that the earth is a perfectly conducting plane, the propagation of MHD waves in a horizontally stratified ionosphere composed of electrons, ions, and neutrals is studied when an arbitrarily oriented plane wave is obliquely incident at the lower edge of the ionosphere. The equations governing the behavior of the field quantities are cast in the "coupled form" suitable for computation by means of a high-speed modern computer, but no numerical results are obtained because of the prohibitive amount of work necessary to solve the equations in the case of oblique incidence.

The velocity of MHD waves is $V_A = B/\sqrt{\mu_0 \rho}$. In a fully ionized gas, ρ is the mass density of the plasma: $\rho = \rho_e + \rho_i$. It is shown that in a partially ionized gas ρ is the effective density of the material responsible for the waves and that for sufficiently low wave frequency and for a weakly ionized plasma, the mass density of the neutrals should be included in ρ or $\rho = \rho_e + \rho_i + \rho_n$.

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I. INTRODUCTION

Recently, the propagation of magnetohydrodynamic (MHD) or hydromagnetic (HM) waves in the ionosphere has been studied extensively by physicists and geophysicists alike. Physicists are interested in this phenomenon as an example of the propagation of very low frequency electromagnetic waves in an inhomogeneous, anisotropic, and bounded medium,[1-6] while geophysicists are interested in it as a possible explanation of geomagnetic micropulsations,[7] giant pulsations,[8] ionospheric noises, the heating of the ionosphere,[9-11] and the propagation of HM pulses that are generated by high-altitude nuclear explosions.[12]

Several conditions must be satisfied before MHD waves can propagate in a partially ionized plasma permeated by an external magnetic field. The more important ones are* (a) the wave frequency ω must be much less than the ion cyclotron frequency ω_i;[13] (b) the phase velocity $V_{ph} = \omega/k$ and the Alfvén speed V_A must both be much less than the velocity of light c; and (c) the conductivity σ must be high enough so that the inequality $\omega/(\mu_o\sigma) \ll V_A^2$ is satisfied.[14] In addition, if the following conditions are also satisfied the theory of HM waves is relatively simple: (d) the static magnetic field is uniform and satisfies the condition for appreciable HM coupling,[15] i.e.,

$$BL\,\sigma\sqrt{(\mu_o/\rho)} \gg 1 \quad ; \qquad (1)$$

(e) the pressure and gravitational forces are negligible compared with the electromagnetic forces; (f) the plasma is in thermal equilibrium and is inviscid; (g) collisions of the charged species with themselves and with each other are negligible compared with collisions between the charged species and the neutrals because the plasma is so slightly ionized[16,17]; (h) the velocity of the neutrals can be neglected; (i) the Alfvén speed V_A is much greater than the sound speed c_o; and (j) the electron collision frequency ν_e is much greater than the electron cyclotron frequency ω_e and the ion collision frequency ν_i is much greater than the ion cyclotron frequency ω_i.

A few comments on the domain of validity and the physical significance of these ten conditions are appropriate. Condition (a) is, in fact, a definition of MHD waves as plasma waves whose

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*See the Glossary for the definition of the more important symbols used in this report.

frequency is much less than the ion cyclotron frequency. When $\omega \ll \omega_i$, the ions cannot be considered to be stationary as they can in the opposite limit when $\omega \gg \omega_i$. When (b) holds, the displacement current may be neglected, in comparison with the conduction current in Ampere's Law $\nabla \times \vec{H} = \vec{J} + (\partial\vec{D}/\partial t) \cong \vec{J}$, and the use of nonrelativistic MHD equations is correct. Physically (b) means that the rest energy density of the plasma $\rho c^2/2$ is much greater than the energy density of the magnetic field $B^2/(2\mu_o)$. The e-folding distance d is the distance in which the amplitude of the MHD waves is attenuated to $e^{-1} \cong 0.368$ of its initial value. It is easily shown[14] that in a plasma of large but finite conductivity

$$d = \frac{2\mu_o \sigma V_A^3}{\omega^2} = \frac{\mu_o \sigma V_A^2 \lambda}{\pi\omega} . \tag{2}$$

In order to ensure that the MHD waves are only slightly damped, condition (c) must hold so that $d/\lambda = \mu_o \sigma V_A^2/\pi\omega \gg 1$. If $d/\lambda \ll 1$, the waves are heavily damped in a fraction of a wavelength, and it makes no sense to talk about wave propagation in the plasma. Likewise, if we assume that the amplitude of the MHD waves varies with time as $\exp(j\omega t)$ and allow ω to be complex, we must require that the time in which the amplitude of the waves is reduced by a factor of $1/e$ of their initial value be much larger than one period of oscillation of the waves. It can be shown without much difficulty that the latter condition requires that the wavelength of the MHD waves be much greater than a critical wavelength $\lambda_c = \pi/\mu_o V_A \sigma$. If (c) is satisfied, then $\lambda \gg \lambda_c$. Condition (d) guarantees that the coupling between the plasma and the static magnetic field is strong, i.e., that the dynamic and magnetic forces are comparable. (If the coupling is weak, we are in the hydrodynamical regime.) The requirement that the seed magnetic field $\vec{B}$ be uniform is not essential, but matters are simplified greatly when it is true. The magnitude of the earth's magnetic field can certainly be assumed to be constant throughout the entire ionosphere; however, its changing direction should be taken into account. Like (a), we can regard conditions (e) and (f) as those necessary for pure MHD waves to be excited in a plasma. The inclusion of pressure, gravitational, and dissipative forces adds nothing important to the picture, but their omission results in great simplification. Conditions (g) and (h) are somewhat more difficult to interpret physically; a good discussion of them is given by Fejer.[18] Investigators in controlled fusion call condition (i) the "low-beta condition," β being the ratio of the sum of the gas and plasma pressure to the magnetic pressure. Thus,

$$\beta = \frac{p}{\frac{B^2}{2\mu_o}} = \frac{\frac{2p}{\rho}}{\frac{B^2}{\mu_o\rho}} \cong \frac{c_o^2}{V_A^2} \quad . \tag{3}$$

The magnetic pressure is approximately 0.01 dyne/cm^2 throughout most of the ionosphere. At 130 km altitude, the sum of the neutral gas and plasma pressures amounts to about 0.01 dyne/cm^2 and decreases rapidly with height so that it is only 10^{-4} dynes/cm^2 in the F_2 layer, or about 300 km. It follows that the assumption that condition (i) holds is reasonable. From (12) it can be easily shown that when condition (j) holds, the use of a scalar conductivity instead of a tensor conductivity is justified. In his original paper, in which the existence of MHD waves is predicted, and in his book, Alfvén used a scalar conductivity. Condition (j) shows when this is permissible.

In this report, we shall assume that all of the ten conditions hold except condition (j). The ionosphere will be assumed to be a horizontally stratified, three-component mixture of electrons, ions, and neutrals whose properties vary with altitude as shown in Table I. From a study of Table I, it can be seen that the ionosphere is a weakly ionized plasma and that condition (g) holds. Further, it is assumed that the earth is a flat, perfectly conducting medium at the low frequencies in which we are interested, viz., $f \leq 1$ cps. The boundary between the bottom of the ionosphere and the upper atmosphere and the top of the ionosphere and the exosphere will be sharp if the wavelength of the MHD waves λ is much larger than the dimensions over which the plasma density decreases to zero. Following Spitzer[19] and estimating the characteristic length L by twice the scale height, we conclude that the lower boundary of the ionosphere may be regarded as sharp, while the top boundary must be considered to be diffuse. The latter conclusions follow from the fact that $L_{80\ km} \cong 10$ km and $L_{550\ km} \cong 200$ km, and for $f = 1$ cps, $\lambda \cong 100$ km. (For estimates of the scale height see reference 20.)

In a recent paper,[11] Francis and Karplus, using the above model and making the above assumptions, studied the propagation of HM waves in the ionosphere. However, they confined themsleves to a vertically incident plane monochromatic wave near 45 degrees geomagnetic lattitude. Here we should like to treat the more general case of an obliquely incident wave and indicate in what sense the problem can be considered to be solved. No numerical results have been obtained with our equations describing oblique incidence because of the large amount of machine time necessary to solve them. It is believed, however, that with the help of the approach detailed in Section II, useful numerical results can be obtained if one has unlimited financial resources.

TABLE I[a]

Temperature T, neutral number density N_n, ion number density N_i, electron number density N_e, average ionic molecular weight W, ion cyclotron frequency ω_i, electron cyclotron frequency ω_e, ion collision frequency ν_i, electron collision frequency ν_e, and plasma frequency f_p versus altitude above the earth's surface. In this table numbers expressed as N^m signify $N \times 10^m$.

Altitude, km	T, °K	N_n Particles/cm^3	$N_i = N_e$ Ions/cm^3	W	ω_i	ω_e	ν_i	ν_e	f_p
80	205	4.30^{14}	1.00^{3}	29.0	1.60^{2}	8.47^{6}	2.05^{5}	3.29^{6}	2.8^{5}
90	225	6.70^{13}	2.50^{4}	29.0	1.60^{2}	8.43^{6}	3.24^{4}	5.42^{5}	1.4^{6}
100	280	1.10^{13}	1.22^{5}	28.4	1.62^{2}	8.39^{6}	5.37^{3}	1.00^{5}	3.2^{6}
120	435	6.20^{11}	1.90^{5}	26.7	1.71^{2}	8.32^{6}	3.12^{2}	7.93^{3}	3.9^{6}
140	620	1.45^{11}	2.35^{5}	25.7	1.76^{2}	8.24^{6}	7.44^{1}	2.66^{3}	4.3^{6}
160	780	5.40^{10}	2.55^{5}	24.7	1.81^{2}	8.16^{6}	2.83^{1}	1.37^{3}	4.5^{6}
180	905	2.50^{10}	2.70^{5}	23.7	1.87^{2}	8.09^{6}	1.33^{1}	8.86^{2}	4.6^{6}
200	995	1.28^{10}	2.75^{5}	23.0	1.91^{2}	8.02^{6}	6.94^{0}	6.46^{2}	4.7^{6}
250	1190	3.54^{9}	3.75^{5}	21.0	2.04^{2}	7.84^{6}	2.01^{0}	3.97^{2}	5.5^{6}
300	1300	1.40^{9}	3.35^{5}	19.5	2.16^{2}	7.66^{6}	8.25^{-1}	3.83^{2}	5.2^{6}
350	1315	6.44^{8}	3.90^{5}	18.4	2.23^{2}	7.49^{6}	3.91^{-1}	4.20^{2}	5.6^{6}
400	1320	3.10^{8}	4.22^{5}	17.4	2.31^{2}	7.33^{6}	1.94^{-1}	4.43^{2}	5.8^{6}
450	1325	1.58^{8}	4.15^{5}	16.8	2.34^{2}	7.17^{6}	1.00^{-1}	4.32^{2}	5.7^{6}
500	1325	8.10^{7}	4.00^{5}	16.2	2.37^{2}	7.01^{6}	5.26^{-2}	4.14^{2}	5.65^{6}
550	1325	4.37^{7}	3.60^{5}	16.1	2.34^{2}	6.86^{6}	2.86^{-2}	3.73^{2}	5.35^{6}

[a] W. E. Francis and R. Karplus, *loc. cit.*

According to the simple theory of HM waves in a fully ionized gas, the velocity of HM waves is[14]

$$V_A = \frac{B}{\sqrt{\mu_o \rho}} \quad . \tag{4}$$

In a partially ionized gas, the question naturally arises: "What is ρ in eq. (4)?" In Section III, it will be shown that ρ is the effective density of the material responsible for the HM waves. If the frequency of the waves is low enough and the ion collision frequency is high enough, the velocity of the ions may be synchronized with the velocity of the neutrals as indicated by (38). This occurs when $Q = 2\omega N_n/(N_i \nu_i)$ is small compared with one. Then $\vec{v}_i \cong \vec{v}_n$, and we must put $\rho = \rho_e + \rho_i + \rho_n$ in (4). As a result, V_A in a weakly ionized plasma can be much less than its value in a fully ionized plasma. On this basis, the rapid decrease of V_A in the D and E layers of the ionosphere, as shown in Fig. 1, is readily understandable. The magnitude of V_A is of considerable physical significance because the e-folding distance d depends on it as shown in (2).

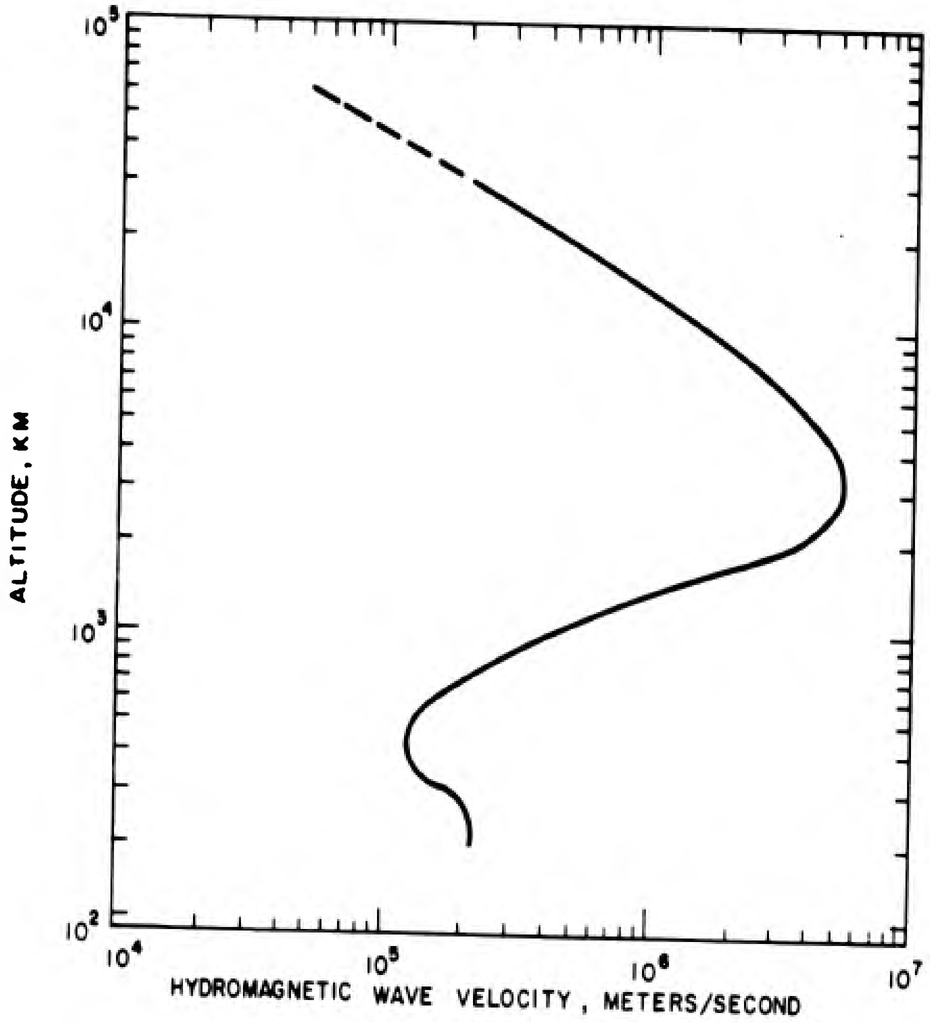

Fig. 1. Hydromagnetic wave velocity as a function of altitude. (From reference 28.)

II. THE PROPAGATION OF PLANE MHD WAVES OBLIQUELY INCIDENT AT THE LOWER EDGE OF THE IONOSPHERE

In this section, the equations governing the propagation of MHD waves in the ionosphere will be enumerated and used to determine, in principle to be sure, the reflection and transmission coefficients of a plane MHD wave obliquely incident on a horizontally stratified ionosphere. Conditions (a) to (i), inclusive, will be assumed to be satisfied throughout the ionosphere. In truth, there are regions of the ionosphere where conditions (c) and (d), as expressed by $d/\lambda \gg 1$ and (1), are not satisfied and there the quasi-hydrodynamical (macroscopic) equations are invalid and must be replaced by Boltzmann's (microscopic) equation. In the interest of simplicity, however, we shall disregard this fact and assume that the quasi-hydrodynamical equations are always valid. We must remember though that the predictions derived from the latter equations are only qualitative.

The relevant equations are Maxwell's equations and the equations of motion of the electrons, ions, and neutrals.[21] (The mks system of units will be employed.)

$$\vec{\nabla} \times \vec{E} = -\mu_0 \dot{\vec{H}} = -j\mu_0\omega\vec{H} \qquad (5)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + k_0\dot{\vec{E}} = \dot{\vec{P}} + k_0\dot{\vec{E}}$$

$$= j\omega\vec{P} + jk_0\omega\vec{E} \qquad , \qquad (6)$$

where

$$\vec{J} = \vec{J}_1 + \vec{J}_2 = N_1e_1\dot{\vec{r}}_1 + N_2e_2\dot{\vec{r}}_2 = \dot{\vec{P}} = \dot{\vec{P}}_1 + \dot{\vec{P}}_2 \qquad (7)$$

$$\vec{P} = N_1e_1\vec{r}_1 + N_2e_2\vec{r}_2 \qquad (8)$$

$$m_1\ddot{\vec{r}}_1 = e_1\left[\vec{E} + (\dot{\vec{r}}_1 \times \mu_0\vec{H})\right] - \Gamma_{12}(\dot{\vec{r}}_1 - \dot{\vec{r}}_2)$$

$$- \Gamma_{13}(\dot{\vec{r}}_1 - \dot{\vec{r}}_3) \qquad (9)$$

$$m_2\ddot{\vec{r}}_2 = e_2\left[\vec{E} + (\dot{\vec{r}}_2 \times \mu_o \vec{H})\right] - \Gamma_{21}(\dot{\vec{r}}_2 - \dot{\vec{r}}_1) - \Gamma_{23}(\dot{\vec{r}}_2 - \dot{\vec{r}}_3) \tag{10}$$

$$m_3\ddot{\vec{r}}_3 = -\Gamma_{31}(\dot{\vec{r}}_3 - \dot{\vec{r}}_1) - \Gamma_{32}(\dot{\vec{r}}_3 - \dot{\vec{r}}_2) \ . \tag{11}$$

where

$$\Gamma_{12} = m_1 m_2 \nu_{12}/(m_1 + m_2) \ , \text{ etc.}$$

In the above equations and throughout this report, the subscripts 1, 2, and 3 refer to electrons, ions, and neutrals, respectively. Thus, $e_1 = -e$, $e_2 = e$ and $e_3 = 0$, where $e = 1.60 \times 10^{-19}$ coulomb. The dot means partial derivative with respect to time, and as we are assuming $d/dt \cong \partial/\partial t$ and a variation with time $\exp(j\omega t)$, the dot means multiplication by $j\omega$. In writing (9) and (10), we have assumed that the omitted terms are negligible by virtue of conditions (e) and (f). Because of conditions (g) and (h), eqs. (9) and (10) can be simplified to

$$m_1\ddot{\vec{r}}_1 = e_1\left[\vec{E} + (\dot{\vec{r}}_1 \times \mu_o \vec{H})\right] - \Gamma_{13}\dot{\vec{r}}_1 \tag{9a}$$

and

$$m_2\ddot{\vec{r}}_2 = e_2\left[\vec{E} + (\dot{\vec{r}}_2 \times \mu_o \vec{H})\right] - \Gamma_{23}\dot{\vec{r}}_2 \ . \tag{10a}$$

It should be noted that in (9) and (10), $\vec{H}$ is the magnetic field of the earth alone, for the magnetic field of the wave is negligible compared with it.[22]

Consider a right-handed rectangular coordinate system with the x-axis pointing vertically upward, the y-axis perpendicular to the magnetic meridian, and the z-axis in the magnetic meridian so that the earth's magnetic field lies in the xz plane. Simple, but rather laborious, algebraic manipulations of (9a), (10a), and (7) then yield[23] $\vec{J} = \vec{J}_1 + \vec{J}_2 = \bar{\bar{\sigma}}_1 \cdot \vec{E} + \bar{\bar{\sigma}}_2 \cdot \vec{E} = \bar{\bar{\sigma}} \cdot \vec{E}$, where

$$\vec{J}_1 = \frac{-j\omega k_o}{a_1(a_1^2 - \zeta_1^2)} \begin{vmatrix} a_1^2 - \zeta_{1x}^2 & ja_1\zeta_{1z} & -\zeta_{1x}\zeta_{1z} \\ -ja_1\zeta_{1z} & a_1^2 & ja_1\zeta_{1x} \\ -\zeta_{1x}\zeta_{1z} & -ja_1\zeta_{1x} & a_1^2 - \zeta_{1z}^2 \end{vmatrix} \vec{E} \;. \tag{12}$$

and a similar equation for $\vec{J}_2$ with the subscript 2 replacing the subscript 1, where

$$\omega_{o_1}^2 = Ne^2/m_1k_o \qquad \omega_1 = e_1B/m_1$$

$$\zeta_1 = \omega\omega_1/\omega_{o_1}^2 \qquad a_1 = \omega^2(1 - j\nu_{13}/\omega)/\omega_{o_1}^2$$

$N_1 \cong N_2 \cong N =$ electron and ion number density. (Quasi-neutrality is assumed.)

(13)

Consider a plane MHD wave which is excited in the ionosphere and which is obliquely incident at the lower edge of the ionosphere-atmosphere interface, but with its plane of incidence in the magnetic meridian or the xz plane as shown in Fig. 2. Call θ_L the angle of incidence at $x = 80$ km and ϕ the angle of refraction. From Snell's Law,

$$\frac{\sin \phi}{\sin \theta_L} = \frac{V_I}{V_{A_L}} = \frac{c}{V_{A_L}} = n_L \quad , \tag{14}$$

or

$$\sin \phi = n_L \sin \theta_L = n \sin \theta \quad . \tag{15}$$

In (15), n is the value of the index of refraction* at the height x in the ionosphere and θ is the angle the wave normal makes with the vertical, as shown in Fig. 2.

As the wave propagates downward, all field quantities will vary as $\exp[j\omega(t - \frac{n}{c}(\sin \theta z - \cos \theta x)]$ in Region II and as $\exp[j(\omega t - k_v (\sin \phi z - \cos \phi x)]$ in Region I, where $k_v = 2\pi f/c$ is the vacuum wavelength. Using (15) $\exp[j\omega(t - \frac{n}{c}(\sin \theta z - \cos \theta x)] = \exp[j\omega (t - \frac{\sin \phi z - n \cos \theta x}{c})]$ and in taking partial derivatives of the field quantities in the ionosphere, we may set $\nabla_y = 0$, $\nabla_z = -\frac{j\omega \sin \phi}{c} = -jk_v \sin \phi = -jk_v \cos Z$, where Z is the angle the refracted ray makes with the z axis. In the atmosphere, we have $\nabla_x = jk_v \cos \phi$, $\nabla_y = 0$ and $\nabla_z = -jk_v \sin \phi$.

If the plane of incidence is not in the plane of the magnetic meridian and the refracted ray makes angles X, Y, and Z with the x-, y-, and z-axis, respectively, then a similar argument shows that

$$\nabla_y = -jk_v \cos Y \quad , \quad \nabla_z = -jk_v \cos Z \qquad \text{in Region II}$$

and (16)

$$\vec{\nabla} = (jk_v \cos X, -jk_v \cos Y, -jk_v \cos Z) \quad \text{in} \quad \text{Region I.}$$

*One of the important effects of the inclusion of collisions is that n is complex and the waves are attenuated as they propagate. In general, θ and θ_L are complex angles.

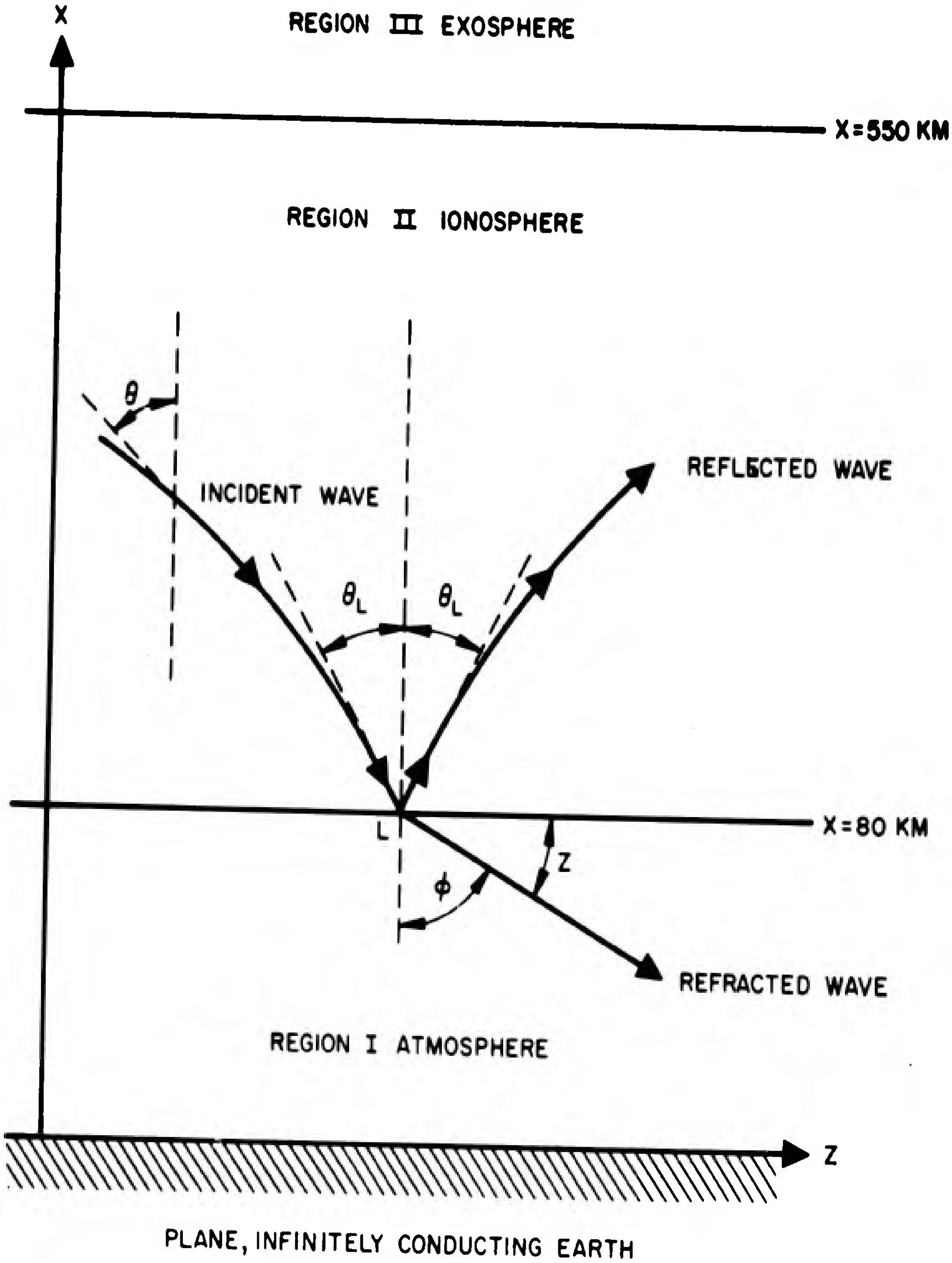

Fig. 2. Plane MHD wave excited in the ionosphere and incident obliquely at the lower edge of the ionosphere and being reflected and refracted. The plane of incidence is in the magnetic meridian. (Not drawn to scale.)

Once the MHD wave has been refracted and emerges in the atmosphere, it is a simple matter to keep track of it by the use of well-known techniques developed for studying the propagation of plane waves in a homogeneous medium. As a useful reference we may cite Chapters 5 and 9 of Stratton.[22] For this reason, in the rest of this section we shall be concerned mainly with studying the history of the incident and reflected MHD wave in the ionosphere.

By using (7), (12), (13), and (16) in Maxwell's equations, (5) and (6), we obtain

$$-jk_v \cos Y\, E_z + jk_v \cos Z\, E_y = -j\omega\mu_o H_x \tag{17}$$

$$-jk_v \cos Z\, E_x - \nabla_x E_z = -j\omega\mu_o H_y \tag{18}$$

$$\nabla_x E_y + jk_v \cos Y\, E_x = -j\omega\mu_o H_z \tag{19}$$

and

$$-jk_v \cos Y\, H_z + jk_v \cos Z\, H_y$$

$$= j\omega k_o \left[\left(1 - \frac{\alpha_1^2 - \zeta_{1x}^2}{\alpha_1(\alpha_1^2 - \zeta_1^2)} - \frac{\alpha_2^2 - \zeta_{2x}^2}{\alpha_2(\alpha_2^2 - \zeta_2^2)} \right) E_x \right.$$

$$- j \left(\frac{\zeta_{1z}}{\alpha_1^2 - \zeta_1^2} + \frac{\zeta_{2z}}{\alpha_2^2 - \zeta_2^2} \right) E_y$$

$$\left. + \left(\frac{\zeta_{1x}\zeta_{1z}}{\alpha_1(\alpha_1^2 - \zeta_1^2)} + \frac{\zeta_{2x}\zeta_{2z}}{\alpha_2(\alpha_2^2 - \zeta_2^2)} \right) E_z \right] \tag{20}$$

$$-jk_v \cos Z\, H_x - \nabla_x H_z = j\omega k_o \left[j\left(\frac{\zeta_{1z}}{a_1^2 - \zeta_1^2} + (2) \right) E_x \right.$$

$$\left. + \left(1 - \frac{a_1}{a_1^2 - \zeta_1^2} - (2) \right) E_y - j\left(\frac{\zeta_{1x}}{a_1^2 - \zeta_1^2} + (2) \right) E_z \right] \tag{21}$$

$$\nabla_x H_y + jk_v \cos Y\, H_x = j\omega k_o \left[\left(\frac{\zeta_{1x}\zeta_{1z}}{a_1^2\,(a_1^2 - \zeta_1^2)} + (2) \right) E_x \right.$$

$$\left. + j\left(\frac{\zeta_{1x}}{a_1^2 - \zeta_1^2} + (2) \right) E_y + \left(1 - \frac{a_1^2 - \zeta_{1z}}{a_1\,(a_1^2 - \zeta_1^2)} - (2) \right) E_z \right] . \tag{22}$$

In (21) through (23), the symbol (2) represents a function of a and ζ exactly like the function of a and ζ preceding the (2), with the subscript 1 replaced by the subscript 2. An inspection of (20) should clarify what is meant. In our treatment, the ions and the electrons are on equal footing and that is why every function of a and ζ with subscript 1 has its counterpart with subscript 2 replacing subscript 1. The 1's in (20) through (23) are the contributions of the displacement current and can be omitted in the MHD approximation.

By virtue of (17) and (20), we can eliminate H_x and E_x in terms of E_y, E_z, H_y, and H_z. Doing this and defining the following quantities, we obtain the desired result, viz., (26).

$$Q_1 = \frac{j\zeta_{1z}}{a_1^2 - \zeta_1^2} + (2) \;, \qquad Q_2 = \frac{j\zeta_{1x}}{a_1^2 - \zeta_1^2} + (2) \;,$$

$$Q_3 = 1 - \frac{a_1}{a_1^2 - \zeta_1^2} - (2) \;, \qquad Q_4 = \frac{\zeta_{1x}\zeta_{1z}}{a_1\,(a_1^2 - \zeta_1^2)} + (2) \tag{23}$$

$$Q_5 = 1 - \frac{a_1^2 - \zeta_{1z}^2}{a_1\,(a_1^2 - \zeta_1^2)} + (2) \;, \qquad Q_6 = \frac{a_1(a_1^2 - \zeta_1^2)}{a_1\,(a_1^2 - \zeta_1^2) - (a_1^2 - \zeta_{1x}^2)} + (2) \;,$$

$$Q_7 = Q_6 Q_4 \;, \qquad Q_8 = Q_6 Q_1 \;,$$

$$Q_9 = \frac{j(a_1 - 1)\zeta_{1x}}{a_1\,(a_1^2 - \zeta_1^2) - (a_1^2 - \zeta_1^2)} + (2) \;, \qquad Q_9 \equiv Q_{9_1} + Q_{9_2} \;,$$

$$Q_{10} = 1 + \frac{ja_1 Q_{9_1}}{\zeta_{1x}} + \frac{ja_2 Q_{9_2}}{\zeta_{2x}} \;,$$

$$Q_{11} = 1 - \frac{1 - j\zeta_{1x} Q_{9_1}}{a_1} - \frac{1 - j\zeta_{2x} Q_{9_2}}{a_2} \qquad (23)$$

$$\bar{\bar{M}} = jk_v \begin{vmatrix} Q_7 \cos Z & -Q_8 \cos Z & Q_6 \cos Z \cos Y & 1 - Q_6 \cos^2 Z \\ Q_7 \cos Y & -Q_8 \cos Y & -1 + Q_6 \cos^2 Y & -Q_6 \cos Z \cos Y \\ Q_9 - \cos Y \cos Z & -Q_{10} + \cos^2 Z & Q_8 \cos Y & -Q_8 \cos Z \\ Q_{11} - \cos^2 Y & Q_9 + \cos Z \cos Y & -Q_7 \cos Y & Q_7 \cos Z \end{vmatrix}$$

(24)

$$\vec{F} = \begin{vmatrix} E_z \\ E_y \\ \eta_o H_z \\ \eta_o H_y \end{vmatrix} , \qquad (25)$$

where

$$\eta_o = \sqrt{(\mu_o/k_o)} = 377 \text{ ohms.}$$

$$\nabla_x \vec{F} = \overline{\overline{M}} \cdot \vec{F} \qquad (26)$$

Eq. (26) is what Clemmow[3, 4] terms the coupled form of the differential equations governing the propagation of MHD waves in the ionosphere. If we are to trace the path of an MHD wave excited in the ionosphere, we must solve (26) subject to the following boundary conditions:

1. $\vec{e}_x \cdot \vec{H} = \vec{e}_x \times \vec{E} = 0 \qquad \text{at } x = 0$

2. $(\vec{e}_x \times \vec{E})$ and $(\vec{e}_x \times \vec{H})$ continuous at $x = 80$ km (27)

3. As x approaches infinity, we have only an outgoing plane wave.

In addition, Table I shows how the properties of the ionosphere vary with altitude or x (see Fig. 2).

Except for very special cases (such as (a) vertical incidence[11] when cos Y = cos Z = 0; (b) oblique incidence and vertical magnetic field when $\zeta_x = \zeta$, $\zeta_y = 0$; and (c) propagation in the magnetic meridian and at the equator when cos Y = 0, cos Z = 1, $\zeta_x = 0$, $\zeta_z = \zeta$), it is well nigh impossible to solve (26). In our problem, we have the additional complication that the geometrics optics approximation so widely used at radio frequencies or microwave frequencies are invalid because the properties of the ionosphere change so rapidly in a wavelength. The last statement can be verified by remembering that $\lambda \cong 100$ km and by studying Table I and Fig. 3.

In references 3 and 4, Clemmow gives a very detailed and clear discussion on how to solve (26) in the three cases mentioned above with the aid of a modern high-speed computing machine such as the IBM 709. The general method will be indicated here, but for details the interested reader is referred to Clemmow's papers.

The matrix $\overline{\overline{M}}$, eq. (24), is a function of x. Presumably it is possible in any small region of the ionosphere to find a local coordinate system in which many of the elements are zero. When this is the case, the solution of (26) is greatly facilitated. Instead of the column vector $\vec{F}$ of (25), imagine that the field quantities are expressed in terms of another column vector $\vec{G}$ related to $\vec{F}$ by the matrix $\overline{\overline{U}}$. Thus,

$$\vec{F} = \overline{\overline{U}} \cdot \vec{G} \quad . \tag{28}$$

Assuming that $\overline{\overline{U}}$ is nonsingular so that it has an inverse $\overline{\overline{U}}^{-1}$, we have, on substituting (28) into (26) and multiplying on the left by $\overline{\overline{U}}^{-1}$,

$$\nabla_x \vec{G} - (\overline{\overline{U}}^{-1}\overline{\overline{M}}\,\overline{\overline{U}}) \cdot \vec{G} = -\overline{\overline{U}}^{-1} (\nabla_x \overline{\overline{U}}) \cdot \vec{G} \quad . \tag{29}$$

If $\overline{\overline{L}} = (\overline{\overline{U}}^{-1}\overline{\overline{M}}\,\overline{\overline{U}})$ is a diagonal matrix, the solution of (29) is considerably easier than the solution of (26). From a well-known theorem in matrix theory, $\overline{\overline{L}}$ is a diagonal matrix, provided the roots of the characteristic equation,[24]

$$\det \left[\overline{\overline{M}} - q\overline{\overline{I}} \right] = 0 \quad , \tag{30}$$

are distinct. Then

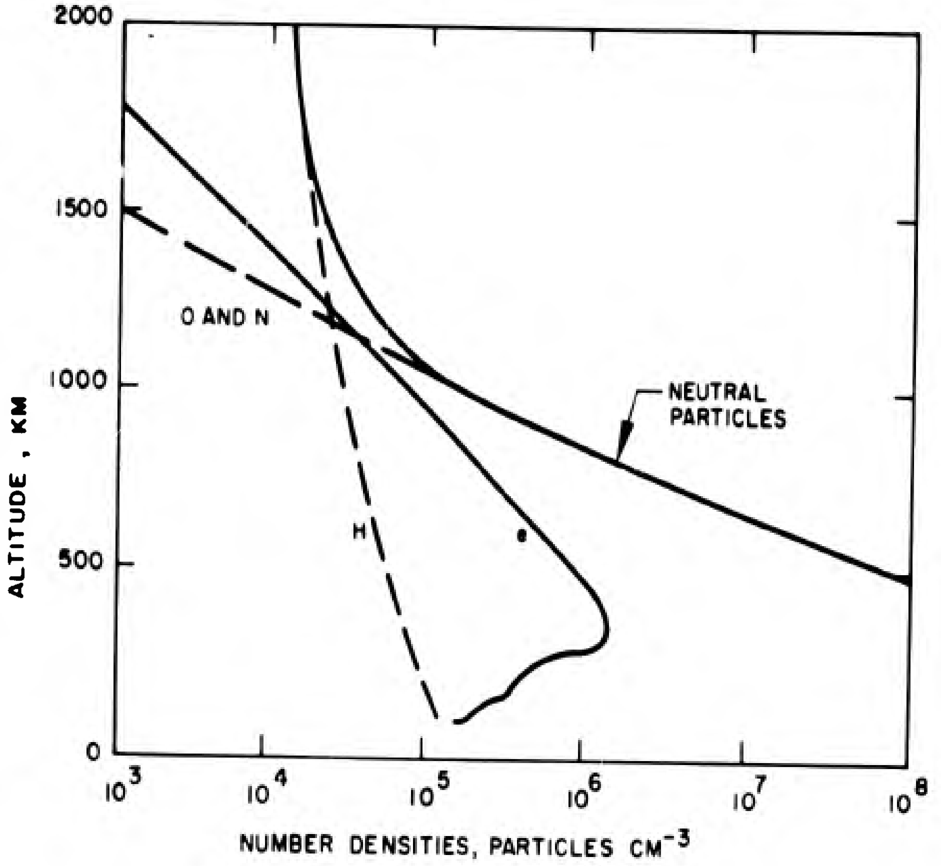

Fig. 3. Electron and neutral particle number densities in the ionosphere and exosphere. (From reference 27.)

$$\bar{\bar{L}} = \begin{vmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{vmatrix} \quad . \tag{31}$$

The equations to be solved are

$$\nabla_x \vec{G} - \bar{\bar{L}} \cdot \vec{G} = -\bar{\bar{U}}^{-1} (\nabla_x \bar{\bar{U}}) \cdot \vec{G} \quad , \tag{32}$$

where

$$\vec{G} = \bar{\bar{U}}^{-1} \cdot \vec{F} \quad . \tag{33}$$

As we mentioned in Section I, Francis and Karplus[11] have solved (32) for vertical incidence with the boundary conditions prescribed by (27), the geometry given by Fig. 2, and the properties of the ionosphere given by Table I. Of the many results they obtained for this simple case, those that interest us most are the (complex) transmission and reflection coefficients as a function of incident angular frequency given by their Table I.[25] Although the equations contained in (32) are much more complicated than those studied by Francis and Karplus, their paper and Budden's[2] are worth careful study if it is deemed important to solve (32).

Now, a system of four linear differential equations is equivalent to a single fourth-order equation which has four independent solutions. Near the top of the ionosphere, two of these are upgoing waves and two are downgoing waves.[24] There are therefore two independent solutions which satisfy the radiation condition of (27). One workable procedure is to start with one "initial" solution and integrate (32) by means of a step-by-step process downward through the ionosphere until a point below the ionosphere is reached. The resulting solution gives the total values of $\vec{G}$ below the ionosphere. It is then separated into an upgoing and a downgoing wave, each being, in general, elliptically polarized. The ratio of the amplitudes of these two waves gives a reflection coefficient, but this applies only to an incident wave of a particular elliptical polarization, which is of no special interest. The process is repeated with the other "initial" solution, and, in general, the resulting upgoing and downgoing waves below the ionsphere have different polarizations from the

first pair. One then takes proper linear combinations of the two solutions such that the incident wave of the combination is of unit amplitude and plane polarized either in the plane of incidence or perpendicular to it. The associated downgoing wave then gives two of the components of the reflection coefficient. The exact procedure which must be followed to determine the reflection coefficients is clearly described in paragraph 8 of reference 2.

III. THE IMPORTANCE OF NEUTRALS IN DETERMINING THE NATURE OF THE MHD WAVES

From Fig. 3 and Table I, it is seen that throughout the entire ionosphere the number density of the neutrals greatly exceeds the number density of the charged particles. An immediate consequence of N_n greatly exceeding N_e and N_i is that in Table I, ν_e and ν_i are the electron and ion collision frequencies with the neutrals alone, for ν_{ee}, ν_{ei}, ν_{ie}, and ν_{ii} are all negligible compared with ν_{en} and ν_{in}.

A very basic and important property of a plasma is its tendency toward electrical neutrality. The theory predicts and measurements confirm that quasi-neutrality or $N_e \cong N_i$ holds in a plasma in equilibrium. A good measure of the correctness of the latter approximation is the smallness of the Debye length ℓ_D compared with all relevant macroscopic dimensions such as the wavelength, mean free path, cyclotron radius, and the dimensions of the system. Although the temperature as a function of altitude is not an accurately known property of the ionosphere, the temperature can be estimated within reasonable limits. It is instructive to note that the maximum value of the Debye length throughout the ionosphere is approximately 0.5 cm.

In a fully ionized plasma the velocity of HM waves is

$$V_A = \frac{B}{\sqrt{\mu_0 \rho}} \quad , \tag{34}$$

where ρ is the mass density of the plasma $\rho = \rho_e + \rho_i$. In a partially ionized plasma, the question naturally arises: "What is ρ in eq. (34)?" From their definitions, it is obvious that if $\nu_i \ll \omega_i$, the motion of the ions is constrained by the magnetic field to be parallel to $\vec{B}$. When ν_{ni} is very large compared with the collision frequencies of the charged species with themselves and with each other, the neutrals in colliding with the ions effectively drag them in the direction parallel to $\vec{B}$. Before the latter statement is true, the mass of the ions and the mass of the neutrals must be approximately equal so that considerable momentum can be exchanged when an ion and a neutral collide. As we can safely assume that $m_i \cong m_n$, it follows that in the lower ionosphere

$$\frac{\vec{B}(\vec{B} \cdot \vec{v}_n)}{B^2} \cong \vec{v}_i \cong \vec{v}_e \quad . \tag{35}$$

Eqs. (9a), (10a), and (11) are the equations of motion of the electrons, ions, and neutrals, respectively. Using

$$\frac{d^2\vec{r}_n}{dt^2} = \frac{d\vec{v}_n}{dt} \cong \frac{\partial \vec{v}_n}{\partial t} = j\omega\vec{v}_n \quad ,$$

we can write (11) as

$$j\omega\vec{v}_n = -\frac{m_e \nu_{ne}}{m_n + m_e} (\vec{v}_n - \vec{v}_e) - \frac{m_i \nu_{ni}}{m_n + m_i} (\vec{v}_n - \vec{v}_i) \quad . \tag{36}$$

The exact expressions for ν_{ne} and ν_{ni}, the collision frequencies between neutrals and electrons and neutrals and ions, respectively, are not known for the temperatures and densities typical of the ionsphere. If we use the expressions given by Chapman and Cowling,[26] we should be safe from great error. Thus, let

$$\nu_{ne} = 2N_e R^2 \left[2\pi K\dot{T}(m_n + m_e)/(m_n m_e) \right]^{1/2} = N_e \nu_{en}/N_n \quad . \tag{37}$$

with a similar expression for ν_{ni}. (In (37) R is the "collision distance" or the distance of closest approach.)

From Table I, it is seen that for altitudes between 80 and 200 km, $\nu_e = 200\,\nu_i$, i.e., $\nu_{ne} = 200\,\nu_{ni}$. When we estimate $m_n = 2 \times 10^4\, m_e$ and remember that $m_n \cong m_i$, the first term on the right-hand side of (36) is negligible compared with the second term. In this approximation and with (35) holding, (36) reduces to

$$\frac{\vec{v}_i}{\vec{v}_n} = 1 + 2j\omega/\nu_{ni} = 1 + 2j\omega N_n/(N_i \nu_i) = 1 + jQ \quad . \tag{38}$$

Depending on the magnitude of Q in (38) relative to one, we can draw the following conclusions: (a) $Q \gg 1$; in this case the neutrals are practically stationary and $\vec{v}_i$ and $\vec{v}_n$ are 90 degrees out of phase. (b) $Q \ll 1$; in this case the motion of the neutrals is almost synchronized with the motion of the ions, i.e.,

$$\vec{v}_i \cong \vec{v}_n \quad . \tag{39}$$

When (39) holds, we must set $\rho = \rho_e + \rho_i + \rho_n$ in (34). In other words, under certain conditions, the presence of the neutrals is important in determining V_A.

GLOSSARY

In this report the mks system of units is used. Some of the more important symbols are listed below.

$\vec{B}$	magnetic induction
c	speed of light in vacuo $\cong 3 \times 10^8$ meters/sec
c_o	speed of sound
d	e-folding distance of MHD waves, eq. (2)
$\vec{D}$	electric displacement
$\vec{E}$	electric intensity, $\vec{D} = k_o \vec{E}$
e	e as a subscript means electrons, magnitude of electron charge
f_p	plasma frequency = $9 \times 10^3 \sqrt{N_e}$ cps
f	frequency of the MHD or EM wave, $2\pi f = \omega$
$\vec{H}$	total magnetic intensity
$\vec{H}_o$	static magnetic intensity, earth's magnetic intensity
$\vec{h}$	perturbed magnetic intensity, $\vec{H} = \vec{H}_o + \vec{h}$, $\lvert\vec{h}\rvert / \lvert\vec{H}_o\rvert \ll 1$
$\bar{\bar{I}}$	unit matrix, eq. (30)
i	i as a subscript means ions

j	the imaginary unit, $j^2 = -1$. Field quantities vary with time as $\exp(j\omega t)$.
$\vec{J}$	current density
k	$2\pi/\lambda$; λ = wavelength of MHD waves
k_o	permittivity of free space = 8.854×10^{-12} farad/meter
K	Boltzmann's constant
L	"characteristic length"; $L_{80\ km}$ is the characteristic length at an altitude of 80 km.
ℓ_D	Debye length = $6.90(T/n_e)^{1/2}$ cm
m_e	electron mass, also designated by m_1
m_i	ion mass, also designated by m_2
N_e	electron number density per cm^3, also designated by N_1
N_i	ion number density, also designated by N_2
N_n	neutral number density, also designated by N_3
$\vec{n}$	unit normal at the boundary of a discontinuity, n as a subscript means neutrals
n	index of refraction = c/V_{ph}
p	pressure

$\vec{P}$	polarization vector, eq. (8)
$\vec{r}$	position vector, (x, y, z)
T	temperature in degrees Kelvin
V_A	Alfvén velocity, eq. (4), Fig. 1
V_{ph}	phase velocity = ω/k
$\vec{v}$	mean fluid velocity in an element of volume ΔV
$\vec{v}_e$	mean electron velocity in an element of volume ΔV
$\vec{v}_i$	mean ion velocity in an element of volume ΔV; $$\vec{v} = \frac{\rho_i \vec{v}_i + \rho_e \vec{v}_e}{\rho_i + \rho_e}$$
W	average ion molecular weight, Table I
x	x coordinate, the altitude
a_1	$\omega^2(1 - j\nu_{13}/\omega)/\omega_{01}^2$, eq. (13)
β	ratio of the sum of the gas and plasma pressure to the magnetic pressure, eq. (3)
ω	angular frequency of the MHD or EM wave, $\omega = 2\pi f$, eq. (7)
$\omega_e = \omega_1$	electron cyclotron frequency = $eB/m_e = 1.8 \times 10^7 B$ (rad/sec, gauss)

$\omega_i = \omega_2$	ion cyclotron frequency = eB/m_i
ω_{o_1}	plasma frequency = $\sqrt{4\pi N_e e^2/m_e}$ = $5.5 \times 10^4 \sqrt{N_e}$ rad/sec
λ	wavelength of MHD waves
μ_o	magnetic permeability = 1.26×10^{-6} henry/meter $\vec{B} = \mu_o \vec{H}$
ρ	mass density
ν_e	electron collision frequency, $\nu_e = \nu_{ee} + \nu_{ei} + \nu_{en}$
ν_i	ion collision frequency
ν_{12}	collision frequency between 1-type particles and 2-type particles; ν_{13} is similarly defined
Γ_{12}	$m_1 m_2 \nu_{12}/(m_1 + m_2)$
σ	scalar conductivity
$\bar{\bar{\sigma}}$	tensor conductivity, eq. (12)
ζ_1	$\omega\omega_1/\omega_{o_1}^2$
ζ_2	$\omega\omega_2/\omega_{o_2}^2$
$\vec{\nabla}$	$(\nabla_x, \nabla_y, \nabla_z)$
1, 2, 3	Used as subscripts 1, 2, 3 refer to electrons, ions, and neutrals, respectively

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